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A Holstein–Primakoff and a Dyson realization for the quantum algebra $U_{\mathbf{q}}[sl(n+1)]$

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Abstract. The known Holstein–Primakoff and Dyson realizations of the Lie algebra $sl(n+1)$, $n = 1, 2, \dots$ in terms of Bose operators are generalized to the class of the quantum algebras $U_q[sl(n+1)]$ for any n. It is shown how the elements of $U_q[sl(n+1)]$ can be expressed via n pairs of Bose creation and annihilation operators.

In this paper we write down an analogue of the Dyson (D) realization and of the Holstein–Primakoff (HP) realization for the quantum algebra $U_q[sl(n+1)]$. Initially both the HP and D realizations were given for $sl(2)$ [1, 2]. The generalization for $gl(n+1)$ is from Okubo [3]. In [3] the elements of $gl(n+1)$ are expressed as functions of n pairs of Bose creation and annihilation operators (CAOs), namely operators $a_1^\pm, a_2^\pm, \dots, a_n^\pm$, which satisfy the known commutation relations

$$[a_i^-, a_j^+] = \delta_{ij} \qquad [a_i^+, a_j^+] = [a_i^-, a_j^-] = 0 \qquad i, j = 1, \dots, n. \tag{1}$$

This realization is 'more economical' than the known Jordan–Schwinger realization, which expresses $gl(n+1)$ via $n+1$ pairs of Bose CAOs.

The motivation in this work stems from the various applications of both the HP and of the D realizations in theoretical physics. Beginning with [1, 2] the HP and D realizations were constantly used in condensed matter physics. Some other early applications can be found in Kittel [4] (more recent results are contained in [5]). For applications in nuclear physics see [6, 7] and references therein, but there are, certainly, several other publications. In view of the importance of the quantum algebras for physics, one could expect that the generalization of the D and HP realizations to the case of quantum algebras may be of interest too. In fact this is the case for the only known q-analogues of the HP realization to date, namely those of $U_q[sl(2)]$ and $U_q[sl(3)]$ [8–15].

Initially we recall the definition of $U_q[sl(n+1)]$ in the sense of Drinfeld [16]. Let $\mathbb{C}[[h]]$ be the complex algebra of the formal power series in the indeterminate h, $q = e^{h/2} \in \mathbb{C}[[h]]$. Then $U_q[sl(n+1)]$ is a Hopf algebra, which is a topologically free $\mathbb{C}[[h]]$ module (complete in the h-adic topology), with generators $\{h_i, e_i, f_i\}_{i=1,\dots,n}$ and

(1) Cartan relations

$$[h_i, e_j] = (2\delta_{ij} - \delta_{i,j-1} - \delta_{i-1,j})e_j \tag{2a}$$

$$[h_i, f_j] = -(2\delta_{ij} - \delta_{i,j-1} - \delta_{i-1,j})f_j \tag{2b}$$

$$[e_i, f_j] = \delta_{ij}\frac{q^{h_i} - \bar{q}^{h_i}}{q - \bar{q}}. \tag{2c}$$

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(2) Serre relations

$$[e_i, e_j] = 0 \qquad [f_i, f_j] = 0 \qquad |i - j| \neq 1 \tag{3a}$$

$$[e_i, [e_i, e_{i\pm1}]_{\bar{q}}]_q = [e_i, [e_i, e_{i\pm1}]_q]_{\bar{q}} = 0 \tag{3b}$$

$$[f_i, [f_i, f_{i\pm1}]_{\bar{q}}]_q = [f_i, [f_i, f_{i\pm1}]_q]_{\bar{q}} = 0. \tag{3c}$$

Throughout $[a, b] = ab - ba$, $[a, b]_x = ab - xba$, $\bar{q} = q^{-1}$. We do not write the other Hopf algebra maps (Δ, ε, S), since we will not use them. They are, certainly, also a part of the definition.

The D and HP realizations are different embeddings of $U_q[sl(n+1)]$ into the Weyl algebra $W(n)$. We define the latter as a topologically free $\mathbb{C}[[h]]$ module and an associative unital algebra with generators $a_1^\pm, \ldots, a_n^\pm$ and relations (1).

Remark. In the physical applications it is often more convenient to consider h and q as complex numbers, $h, q \in \mathbb{C}$. Then all of our considerations remain true providing that q is not a root of 1. The replacement of $q \in \mathbb{C}[[h]]$ with a number corresponds to a factorization of $U_q[sl(n+1)]$ and $W(n)$ with respect to the ideals generated by the relation $q = number$. The factor-algebras $U_q[sl(n+1)]$ and $W(n)$ are complex associative algebras. However, the completion in the h-adic topology has left a relevant trace: after the factorization the elements of $U_q[sl(n+1)]$ and of $W(n)$ are not only polynomials of their generators. In particular the functions of the CAOs, which appear in both the D and HP realizations (see (4) and (10) below) are well defined as elements from $W(n)$.

We are now ready to state our main results. Let $[x] = \frac{q^x - \bar{q}^x}{q - \bar{q}}$, $N_i = a_i^+ a_i^-$ and $N = N_1 + \ldots + N_n$.

Proposition 1 (D realization). The linear map $\varphi : U_q[sl(n+1)] \to W(n)$, defined on the generators as

$$\varphi(h_1) = p - N - N_1 \qquad \varphi(h_i) = N_{i-1} - N_i \qquad i = 2, 3, \ldots, n \tag{4a}$$

$$\varphi(e_1) = \frac{[N_1 + 1]}{N_1 + 1}[p - N]b_1^-, \qquad \varphi(e_i) = \frac{[N_i + 1]}{N_i + 1} b_i^- b_{i-1}^+ \qquad i = 2, \ldots, n \tag{4b}$$

$$\varphi(f_1) = b_1^+ \qquad \varphi(f_i) = \frac{[N_{i-1} + 1]}{N_{i-1} + 1} b_i^+ b_{i-1}^- \qquad i = 2, \ldots, n \tag{4c}$$

is a morphism of $U_q[sl(n+1)]$ into $W(n)$ for any $p \in \mathbb{C}$.

The proof is straightforward. In the intermediate computations the following relation is useful:

$$f(N_1, \ldots, N_i, \ldots, N_n) a_j^\pm = a_j^\pm f(N_1 \pm \delta_{1j}, \ldots, N_i \pm \delta_{ij}, \ldots, N_n \pm \delta_{nj}) \tag{5}$$

where $f(N_1, \ldots, N_i, \ldots, N_n) \in W(n)$ is a function of the number operators $N_1, \ldots, N_i, \ldots, N_n$.

We have derived the D realization (4) on the ground of an alternative to the Chevalley definition of $U_q[sl(n+1)]$ [17]. This derivation together with the expressions for (the analogues of) all Cartan–Weyl generators via CAOs will be given elsewhere.

Similarly as for $sl(n+1)$, the D realization defines an infinite-dimensional representation of $U_q[sl(n+1)]$ in the Fock space $\mathcal{F}(n)$ with an orthonormalized basis

$$|l\rangle \equiv |l_1, \ldots, l_n\rangle = \frac{(a_1^+)^{l_1} \ldots (a_n^+)^{l_n}}{\sqrt{l_1! \ldots l_n!}} |0\rangle, \qquad l_1, \ldots, l_n = 0, 1, 2, \ldots. \tag{6}$$

If p is a positive integer, $p \in \mathbb{N}$, the representation is indecomposible: the subspace

$$\mathcal{F}_1(p;n) = \text{lin.env.}\{|l_1,\ldots,l_n\rangle | l_1+\ldots+l_n > p\} \tag{7}$$

is an invariant subspace, whereas its orthogonal compliment

$$\mathcal{F}_0(p;n) = \text{lin.env.}\{|l_1,\ldots,l_n\rangle | l_1+\ldots+l_n \leqslant p\} \tag{8}$$

is not an invariant subspace. If $p \notin \mathbb{N}$, the representation is irreducible. In all cases, however, and this is the disadvantage of the D realization, the representation of $U_q[sl(n+1)]$ in $\mathcal{F}(n)$ is not unitarizable with respect to the antilinear anti-involution $\omega : U_q[sl(n+1)] \to U_q[sl(n+1)]$, defined on the generators as

$$\omega(h_i) = h_i \qquad \omega(e_i) = f_i \qquad i = 1,\ldots,n. \tag{9}$$

In order to 'cure' this disadvantage we now introduce the HP realization.

Proposition 2 (HP realization). The linear map $\pi : U_q[sl(n+1)] \to W(n)$, defined on the generators as

$$\pi(h_1) = p - N - N_1 \qquad \pi(h_i) = N_{i-1} - N_i \qquad i = 2,3,\ldots,n \tag{10a}$$

$$\pi(e_1) = \sqrt{\frac{[N_1+1]}{N_1+1}[p-N]}a_1^- \qquad \pi(e_i) = \sqrt{\frac{[N_{i-1}]}{N_{i-1}}\frac{[N_i+1]}{N_i+1}}a_i^- a_{i-1}^+ \qquad i = 2,3,\ldots,n \tag{10b}$$

$$\pi(f_1) = \sqrt{\frac{[N_1]}{N_1}[p-N+1]}a_1^+ \qquad \pi(f_i) = \sqrt{\frac{[N_{i-1}+1]}{N_{i-1}+1}\frac{[N_i]}{N_i}}a_i^+ a_{i-1}^- \qquad i = 2,3,\ldots,n \tag{10c}$$

is a morphism of $U_q[sl(n+1)]$ into $W(n)$ for any $p \in \mathbb{C}$. If $p \in \mathbb{N}$, then $\mathcal{F}_0(p;n)$ and $\mathcal{F}_1(p;n)$ are invariant subspaces; $\mathcal{F}_0(p;n)$ carries a finite-dimensional irreducible representation; it is unitarizable with respect to the anti-involution (9) and the metric is defined with the orthonormed basis (6), provided $q > 0$.

The proof is straightforward: the verification of the defining relations (2) and (3) can be carried out on a purely algebraic level. The circumstance that $\mathcal{F}(n)$ is a direct sum of its invariant subspaces $\mathcal{F}_0(p;n)$ and $\mathcal{F}_1(p;n)$ is due to the the factor $\sqrt{[p-N]}$ in (10*b*) and $\sqrt{[p-N+1]}$ in (10*c*). If $q > 0$, then ((,) denotes the scalar product)

$$(\pi(h_i)|l\rangle, |l'\rangle) = (|l\rangle, \pi(h_i)|l'\rangle) \qquad (\pi(e_i)|l\rangle, |l'\rangle) = (|l\rangle, \pi(f_i)|l'\rangle)$$
$$\forall |l\rangle, |l'\rangle \in \mathcal{F}_0(p;n) \qquad i = 1,\ldots,n.$$

Therefore the representation of $U_q[sl(n+1)]$ in $\mathcal{F}_0(p;n)$ is unitarizable.

Let us note that the HP realization (10) of $U_q[sl(n+1)]$ can also be easily expressed in terms of deformed oscillator operators $\tilde{a}_i^\pm$, $\tilde{N}_i$, $i = 1,\ldots,n$, namely operators which satisfy the relations [18–20]:

$$[\tilde{a}_i^-, \tilde{a}_j^+]_q = \delta_{ij} q^{-\tilde{N}_i} \qquad [\tilde{N}_i, \tilde{a}_j^\pm] = \pm\delta_{ij}\tilde{a}_j^\pm \qquad [\tilde{a}_i^\pm, \tilde{a}_k^\pm] = [\tilde{N}_i, \tilde{N}_k] = 0 \qquad \text{i} \neq k. \tag{11}$$

From (10) and the relations between the deformed and the nondeformed operators [21]

$$\tilde{a}_i^- = \sqrt{\frac{[N_i+1]}{N_i+1}}a_i^- \qquad \tilde{a}_i^+ = \sqrt{\frac{[N_i]}{N_i}}a_i^- \qquad \tilde{N}_i = N_i \tag{12}$$

one obtains a q-analogue of the HP realization for any n:

$$\pi(h_1) = p - \tilde{N} - \tilde{N}_1 \qquad \pi(h_i) = \tilde{N}_{i-1} - \tilde{N}_i \qquad i = 2, 3, \ldots, n \tag{13a}$$

$$\pi(e_1) = \sqrt{[p - \tilde{N}]}\tilde{a}_1^- \qquad \pi(e_i) = \tilde{a}_i^- \tilde{a}_{i-1}^+ \qquad i = 2, 3, \ldots, n \tag{13b}$$

$$\pi(f_1) = \sqrt{[p - \tilde{N} + 1]}\tilde{a}_1^+ \qquad \pi(f_i) = \tilde{a}_i^+ \tilde{a}_{i-1}^- \qquad i = 2, 3, \ldots, n. \tag{13c}$$

To the best of our knowledge such q-deformed analogues of HP realizations are only available so far for $U_q[sl(2)]$ [8–14] and for $U_q[sl(3)]$ [15].

Finally, by adding to the generators of $U_q[sl(n+1)]$ an additional central element I, and setting $\varphi(I) = \pi(I) = p$, one obtains D and HP realizations of $U_q[gl(n+1)]$.

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